MATH-5B TEST 2 v4 (7.1-7.5 and 7.8)

Spring 2024

100 points **NAME:**__________________________

- **Phones must be turned OFF and put away. Any visible phone (smart watch, headphones, ipad etc.) will result in a grade F . Hands must remain in view during the exam.**
- **No scratch paper.**
- **No credit will be given for solutions if work is not shown.**
- **I expect clear and legible presentations with words of explanation.**

YOU MAY USE ONE PAGE OF NOTES.

Fill in the blanks. (3 points each)

(1) $\int \sin^2 x \, dx =$ $\frac{1}{2}x - \frac{1}{4}\sin 2x + C$

(2) $\int \cos\left(\frac{1}{2}x\right) dx =$ $2\sin\left(\frac{1}{2}x\right) + C$

(3) $\int \frac{1}{5x+4} \, dx =$ $\frac{1}{5}\ln|5x+4| + C$

FOR PROBLEMS 4 - 12, INTEGRATE AND SIMPLIFY (9 pts each)

You may not use any integration formulas other than those covered in class.

(4) $\int \tan x \sec^3 x \, dx$

$= \int \tan x \sec x \sec^2 x \, dx$ $\quad u = \sec x$, $du = \sec x \tan x \, dx$

$= \int u^2 \, du$

$= \frac{1}{3}u^3 + C$

$= \frac{1}{3}\sec^3 x + C$

Remember - you can always check by differentiation.

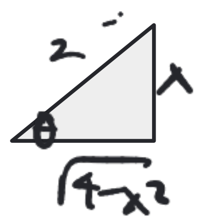
$\frac{d}{dx}\left(\frac{1}{3}\sec^3 x\right) = \sec^2 x \frac{d}{dx}(\sec x) = \sec^2 x \sec x \tan x$

$= \sec^3 x \tan x$ ✓

(5) $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\,dx$

$x = 2\sin\theta$

$dx = 2\cos\theta\,d\theta$



$$= \int \frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}}\, 2\cos\theta\, d\theta$$

$$= \int 4\sin^2\theta\, d\theta$$

$$= 4\left(\tfrac{1}{2}\theta - \tfrac{1}{2}\sin\theta\cos\theta\right) + C$$

$$= 2\theta - 2\sin\theta\cos\theta + C$$

$$= 2\sin^{-1}\frac{x}{2} - 2\,\frac{x}{2}\,\frac{\sqrt{4-x^2}}{2} + C$$

$$= 2\sin^{-1}\frac{x}{2} - \frac{x\sqrt{4-x^2}}{2} + C$$

(6) $\displaystyle\int \frac{e^{1/x}}{x^3}\,dx$

$u = 1/x$

$du = -\frac{1}{x^2}dx$

$$\int e^{\frac{1}{x}}\, \frac{1}{x}\cdot\frac{1}{x^3}\,dx$$

$$-\int u e^u\, du$$

$v = u \quad dv = e^u du$

$dv = du \quad v = e^u$

$$-\left(ue^u - \int e^u du\right) + c$$

$$-ue^u + e^u + c$$

$$-\frac{e^{1/x}}{x} + e^{1/x} + c$$

This was a HW problem

(7) $\int x\sqrt{7-x^2}\,dx$

$u = 7 - x^2$

$du = -2x\,dx$

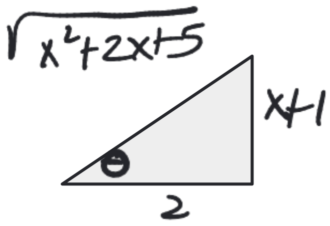
$-\frac{1}{2}\int u^{1/2}\,du$

$-\frac{1}{3}\;u^{3/2} + C$

$-\frac{1}{3}(7-x^2)^{3/2} + C$

Easiest as a u-substitution

(8) $\int \frac{1}{\sqrt{x^2+2x+5}}\,dx$



$= \int \frac{1}{\sqrt{(x+1)^2+4}}\,dx$

$x+1 = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

$= \int \frac{1}{\sqrt{4\tan^2\theta+4}}\,2\sec^2\theta\,d\theta$

$= \int \sec\theta\,d\theta$

$= \ln|\sec\theta + \tan\theta| + C$

$= \ln\left|\frac{\sqrt{x^2+2x+5}}{2} + \frac{x+1}{2}\right| + C$

(This was a HW problem)

(9) $\int \frac{1}{1+\sqrt[3]{x}}\,dx$

$u = x^{1/3}$

$u^3 = x$

$3u^2\,du = dx$

This was on the sample test.

$$\int \frac{3u^2}{1+u}\,du \longrightarrow$$

divide

$$\begin{array}{r|l} & 3u-3 \\ \hline u+1 & 3u^2 \\ & -(3u^2+3u) \\ \hline & -3u \\ & -(-3u-3) \\ \hline & 3 \end{array}$$

$$= \int \left(3u - 3 + \frac{3}{u+1}\right) du$$

$$= \frac{3}{2}u^2 - 3u + 3\ln|u| + C$$

$$= \frac{3}{2}x^{2/3} - 3x^{1/3} + 3\ln|x^{1/3}+1| + C$$

(10) $\int \frac{4x^2-2x}{(x^2+1)(x-1)}dx = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+(Bx+C)(x-1)}{(x-1)(x^2+1)}$

Note:. If your solution gets overly arithmetically complicated, you have probably made a mistake.

$* \quad 4x^2-2x = A(x^2+1)+(Bx+C)(x-1)$

$4x^2-2x = A(x^2+1) + Bx^2+Cx-Bx-C$

$4x^2-2x = (A+B)x^2+(C-B)x+A-C$

$A+B=4$
$C-B=-2$
$A-C=0$

Also, short cut:
if $x=1$
$* \Rightarrow \quad 2=2A$
$A=1$
$\Rightarrow B=3,\ C=1$

$\int \frac{4x^2-2x}{(x^2+1)(x-1)}dx = \int \left(\frac{1}{x-1} + \frac{3x+1}{x^2+1}\right)dx$

$= \int \frac{1}{x-1}dx + \int \frac{3x}{x^2+1}dx + \int \frac{1}{x^2+1}dx$

$u=x-1$ $\qquad t=x^2+1$
$\qquad dt=2x\,dx$

$= \int \frac{1}{u}du + \frac{3}{2}\int \frac{1}{t}d \quad + \tan^{-1}x + C$

$= \ln|u| + \frac{3}{2}\ln|t| + \tan^{-1}x + C$

$$\boxed{= \ln|x-1| + \frac{3}{2}\ln(x^2+1) + \tan^{-1}x + C}$$

Find the value of the following **improper** integrals. Be sure to use all appropriate notation and show **all** steps. (If problem involves an indeterminate limit, SHOW ALL STEPS.) (9 points each)

(11)

$$\int_0^\infty \frac{x}{x^4+1}\,dx$$

$$\int \frac{x}{x^4+1}\,dx \qquad u = x^2 \quad du = 2x\,dx$$

$$\frac{1}{2}\int \frac{du}{u^2+1} = \frac{1}{2}\tan^{-1}u + c$$

$$= \frac{1}{2}\tan^{-1}(x^2) + C$$

$$= \lim_{t\to\infty}\int_0^t \frac{x}{x^4+1}\,dx$$

$$= \lim_{t\to\infty} \frac{1}{2}\tan^{-1}(x^2)\Big]_0^t$$

$$= \lim_{t\to\infty} \frac{1}{2}\tan^{-1}(t^2)$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$$

(12) $\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$

$$\int \frac{\ln x}{\sqrt{x}}\,dx = \int x^{-1/2}\ln x\,dx$$

IBP $\quad u = \ln x \quad dv = x^{-1/2}dx$

$\quad du = 1/x\,dx \quad v = 2x^{1/2}$

$$\int \frac{\ln x}{\sqrt{x}}\,dx = 2\sqrt{x}\ln x - \int 2x^{-1/2}dx$$

$$= 2\sqrt{x}\ln x - 4x^{1/2} + C$$

$$\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{\ln x}{\sqrt{x}}\,dx$$

$$= \lim_{t\to 0^+} 2\sqrt{x}\ln x - 4\sqrt{x}\Big]_t^1$$

$$= \lim_{t\to 0^+} -4 - (2\sqrt{t}\ln t - 4\sqrt{t})$$

$$= \lim_{t\to 0^+} -4 - 2\sqrt{t}\ln t = -4 - \lim_{t\to 0^+}\sqrt{t}\ln t = -4$$

indeterminate $0\cdot -\infty$

$$\lim_{t\to 0^+}\sqrt{t}\ln t = \lim_{t\to 0^+}\frac{\ln t}{t^{-1/2}}$$

$$\stackrel{L}{=} \lim_{t\to 0^+}\frac{\frac{1}{t}}{-\frac{1}{2}t^{-3/2}} = \lim_{t\to 0^+} -2\sqrt{t} = 0$$

0

(13) $[0, 1]$ (10 points)

Removed

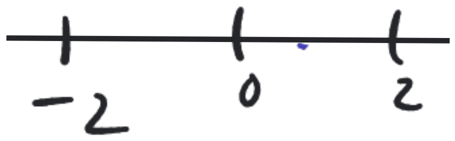
over $[0,2]$

Find the area of the region enclosed by $f(x) = x^3\sqrt{4-x^2}$, and the x axis

where does the graph intersect the x-axis

$$0 = x^3\sqrt{4-x^2}$$

$$x = 0, 2, -2$$



f(x) is odd, symmetry w.r.t origin
so we can find the area on

$$\int_0^2 x^3\sqrt{4-x^2}\,dx \qquad u = 4-x^2 \qquad x^2 = 4-u$$
$$du = -2x\,dx$$

$$-\frac{1}{2}\int_4^0 (4-u)\sqrt{u}\,du = \frac{1}{2}\int_0^4 (4u^{1/2} - u^{3/2})\,du$$

$$\frac{1}{2}\left(\frac{8}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right)\Big]_0^4$$

$$\frac{4}{3}u^{3/2} - \frac{1}{5}u^{5/2}\Big]_0^4$$

$$\frac{4}{3}\cdot 8 - \frac{1}{5}\cdot 32 = \frac{32}{3} - \frac{32}{5} = \frac{256}{15}$$